ICASE REPORT

ON THE MATHEMATICAL CONDITIONS FOR THE EXISTENCE OF PERIODIC FLUCTUATIONS IN NONUNIFORM MEDIA

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ON THE MATHEMATICAL CONDITIONS FOR THE EXISTENCE OF PERIODIC FLUCTUATIONS IN NONUNIFORM MEDIA

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ABSTRACT

In many areas of mathematical physics where one is interested in the propagation of waves through nonuniform media, it is often assumed that periodic excitations result in periodic responses. This assumption is examined by rigorously investigating the existence of periodic solutions of linear hyperbolic differential equations whose coefficients vary with position and whose solution must satisfy periodic boundary or source data. It is shown that the nature of the coefficients of the undifferentiated terms of the differential system is crucial in determining whether or not the solution is periodic. In physical applications, these coefficients usually depend on the gradients of media properties as well as on the media properties themselves. In particular, it is shown that for a general hyperbolic system of two equations in one space dimension, the solution is not periodic. Moreover, this can remain true even if the media gradients are assumed small. However, if the media gradients vanish, or if they vanish except for a bounded region of space, the solution is shown to be periodic for a large enough time. Furthermore, if these gradients vanish asymptotically at large distances, then the disturbances will be asymptotically periodic for increasing time. Special attention is given to the propagation of infinitesimal pressure disturbances through nonuniform steady flows of a lossless fluid.

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1. INTRODUCTION

Historically, the mathematical treatment of problems in linear wave propagation has often been aided by the assumption that periodic excitations result in periodic responses (see, for instance, [1]-[7]). More precisely, it is assumed that *if the excitations (caused by sources or at boundaries) are periodic with a frequency* ω *, then for a large enough time, the response will also be periodic in time with a frequency* ω . The response frequency ω is supposedly independent of position. However, the time one has to wait in order to observe the periodicity may depend on position.

Of particular interest here are those phenomena which are governed by a system of linear hyperbolic partial differential equations whose coefficients may vary with the space coordinates. Then the periodicity assumption results in a great simplification: the time dependent linear hyperbolic system reduces to a time independent linear elliptic system, i.e. the number of independent variables have been reduced by one. It is the purpose of this work to examine the validity of this periodicity assumption.

Certainly, for the propagation of waves in uniform media, the periodicity assumption has been used with great success. Indeed, it can be shown (see section 3 below) that for vanishing initial conditions, the periodicity of the solution is not an assumption for uniform media, but is a mathematical reality. When waves are propagating through a nonuniform media, the coefficients of the governing differential system depend not only on the properties of the media, but also on their gradients. It is usually the case that media gradients enter only into the coefficients of terms involving undifferentiated values of the dependent variables. (Hereafter, we will refer to these coefficients as "lower order coefficients".)

It is shown below that it is precisely these coefficients which are crucial in determining whether or not the solution of the governing differential system is periodic. In section 3 we examine the periodicity question for the general case of the lower order coefficients being nonzero and of arbitrary magnitude throughout the range of validity of the differential system stated in section 2. We then examine three important physically motivated special cases. First, in section 4, we assume that the lower order coefficients are small (in some sense). Then, in section 5, we assume that these coefficients either vanish everywhere except for a region near the origin, where their magnitude may be arbitrary. In section 5 we also consider the case of the lower order coefficients vanishing asymptotically at large distances.

In section 6 we study the implications to physical phenomena which can be described by equations such as those presented in section 2. A detailed account is given for one such phenomena, i.e. the propagation of infinitesimal pressure fluctuations through nonuniform steady flows of a lossless fluid.

2. THE MATHEMATICAL PROBLEM

As a prototype governing system we examine a system of two equations with one space-like independent variable which without loss of generality may be written as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix}_t + \begin{pmatrix} \lambda(x) & 0 \\ 0 & -\sigma(x) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}_x + \begin{pmatrix} \bar{A}(x) & \bar{B}(x) \\ \bar{C}(x) & \bar{D}(x) \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = 0 \ . \tag{1}$$

The mathematical problem to be considered is whether or not a system such as (1) has periodic solutions (for some range of x and t) for some set of given initial and (periodic) boundary and/or source conditions. Restriction is now made to those problems for which λ and σ are positive. The analysis of this work can be carried out in a similar manner for the "supersonic" case, i.e. $\lambda>0$ and $\sigma<0$.

The system (1) can be considered valid for either $-\infty<x<\infty$ or $x>0$. In the first case the driver is a periodic source which introduces into the right hand side of (1) the vector

$$\begin{pmatrix} m(x) \\ n(x) \end{pmatrix} \exp(i\omega t) \quad . \tag{2}$$

In particular, a point source located at x=a would yield $m(x)=n(x)=q\delta(x-a)$ where q is a constant. For the semi-infinite region $x>0$, the driver may be a source such as (2) with an additional boundary condition at x=0. On the other hand, the boundary condition itself may drive the problem so that (1) remains homogeneous. In this work we present results for the boundary condition

$$\phi(x=0) = q \exp(i\omega t) \quad . \tag{3}$$

The general results obtained are equally valid for problems driven by sources or by more general boundary conditions of the type

$$\phi(x=0) + b\psi(x=0) = q \exp(i\omega t) \quad .$$

Specific results for these problems can be deduced in a similar manner as those obtained below for the boundary condition (3).

Finally, in order to close the mathematical description of the problem, initial conditions must be specified. In general, physically correct initial conditions are not easy to define, and in fact, one of the main reasons for looking for periodic solutions is that these would be independent of the initial data. For the purposes of this work, the general initial conditions

$$\phi(t=0) = f(x) \quad \text{and} \quad \psi(t=0) = g(x) \tag{4}$$

will be used. However, the very reasonable and perhaps physically applicable initial condition $\phi=\psi=0$ will be kept in mind. Vanishing initial data implies that there is no disturbance for $t<0$, and that the boundary or source disturbance is started at $t=0$.

Because of the linearity of the differential system and side conditions, the problem of sources and/or boundaries and initial data can be treated separately; for example, for a boundary/initial value problem with no source, if $\phi=\phi_1 + \phi_2$, then ϕ_2 can satisfy homogeneous boundary data with non-homogeneous initial data, and vice versa for ϕ_1.

With the introduction of the characteristic independent variables

$$\alpha = t + \int_0^x \frac{d\xi}{\sigma(\xi)} \quad \text{and} \quad \beta = t - \int_0^x \frac{d\xi}{\lambda(\xi)} \tag{5}$$

the system (1) becomes

$$\begin{aligned} &\phi_\alpha + A(\alpha-\beta)\phi + B(\alpha-\beta)\psi = 0 \\ &\psi_\beta + C(\alpha-\beta)\phi + D(\alpha-\beta)\psi = 0 \end{aligned} \tag{6}$$

since from (5) x is a function of $(\alpha-\beta)$ only. In (6)

$$(\sigma+\lambda)A = \overline{A}\sigma \ , \quad (\sigma+\lambda)B = \overline{B}\sigma \ , \quad (\sigma+\lambda)C = \overline{C}\lambda \ , \quad \text{and } (\sigma+\lambda)D = \overline{D}\lambda \ .$$

3. NONPERIODICITY FOR THE GENERAL SYSTEM

Before examining the nature of the solutions of the system (6), the side conditions must be transformed into the characteristic plane. From (5) the boundary line $x=0$ becomes the line $\alpha=\beta$, and the initial line $t=0$ becomes a curve which can be given parametrically as $\alpha=h_1(x)$ and $\beta=h_2(x)$. The origin $x=t=0$ transforms to $\alpha=\beta=0$.

In this section a detailed proof is given for the non-periodicity of solutions of the system (6) with the side conditions (3) and (4) which transform to

$$\begin{aligned} &\phi(\alpha,\beta) = \exp(i\omega t) \quad \text{for} \quad \alpha=\beta=t \\ &\left.\begin{aligned} \phi(\alpha,\beta) &= f(x) \\ \psi(\alpha,\beta) &= g(x) \end{aligned}\right\} \quad \text{for} \quad \left\{\begin{aligned} &\alpha=h_1(x) \\ &\beta=h_2(x) \end{aligned}\right. . \end{aligned} \tag{7}$$

The results of this section will hold for any other combination of side conditions discussed in section 2, although for side conditions other than (7) no detailed proofs will be presented in this work.

As indicated in section 2, two sub-problems will be considered. First let

$$\phi_1(\alpha,\beta) = \exp(i\omega t) \qquad \text{for } \alpha = \beta \quad (x=0)$$

and (8)

$$\phi_1(\alpha,\beta) = \psi_1(\alpha,\beta) = 0 \qquad \text{for } t = 0$$

where ϕ_1,ψ_1 are solutions of the system (6). Then let

$$\begin{aligned} &\phi_2(\alpha,\beta) = 0 && \text{for} \quad x = 0 \\ &\phi_2(\alpha,\beta) = f(x) \text{ and } \psi_2(\alpha,\beta) = g(x) && \text{for} \quad t = 0 \end{aligned} \tag{9}$$

where ϕ_2,ψ_2 are also solutions of (6). Then $\phi=\phi_1+\phi_2$ and $\psi=\psi_1+\psi_2$ will satisfy (6) and the side conditions (7). Note that if the initial conditions vanish, i.e. f=g=0, then $\phi_2=0$ so that $\phi=\phi_1$. The nonperiodicity of solutions is now proven by contradiction.

It is clear that due to the vanishing initial data for ϕ_1 and ψ_1, that $\phi_1=\psi_1=0$ ahead of the right running characteristic passing through the origin

$$\beta = t - \int_0^x \frac{d\xi}{\lambda(\xi)} = 0 \quad .$$

Then, on this curve ($\beta=0$) the side condition $\psi_1=0$ may be imposed. Now, solutions are sought in the region $0<\beta<\alpha$, $\alpha>0$ (see Figure 1).

The first equation of the system (6) may be interpreted as an ordinary differential equation for ϕ as a function of α with β playing the role of a parameter. The solution ϕ_1 to this ordinary differential equation is subject to the "initial condition" $\phi_1(\alpha=\beta) = \exp\,(i\omega t)$. Then, formally

$$\phi_1(\alpha,\beta)=\exp\left\{-\int_\beta^\alpha A(\xi-\beta)d\xi\right\}\exp\left\{i\omega\beta\right\}-\int_\beta^\alpha d\xi B(\xi-\beta)\psi_1(\xi,\beta)\exp\left\{\int_\alpha^\xi A(\xi'-\beta)d\xi'\right\}. \tag{10}$$

Likewise, the second equation of (6) can be thought of as an ordinary differential equation for ψ_1 as a function of β with the "initial condition" $\psi_1(\beta=0)=0$ where now α can be considered to be a parameter. Then formally,

$$\psi_1(\alpha,\beta) = -\int_0^\beta d\eta C(\alpha-\eta)\phi_1(\alpha,\eta)\exp\left\{\int_\beta^\eta D(\alpha-\eta')d\eta'\right\} \quad . \tag{11}$$

Substituting (11) into (10) yields the integral equation for ϕ_1

$$\phi_1(\alpha,\beta) = \exp\left\{\int_\alpha^\beta A(\xi-\beta)d\xi\right\}\exp\left\{i\omega\beta\right\}$$
$$+\int_\beta^\alpha d\xi B(\xi-\beta)\exp\left\{\int_\alpha^\xi A(\xi'-\beta)d\xi'\right\}\int_0^\beta d\eta C(\xi-\eta)\phi_1(\xi,\eta)\exp\left\{\int_\beta^\eta D(\xi-\eta')\,d\eta'\right\} . \tag{12}$$

The region of integration for the double integral appearing in (12) is the quadrilateral $\beta<\xi<\alpha$, $0<\eta<\beta$.

From (5) it is obvious that the characteristic curves of the same family are equally spaced in time in the sense that

$$\alpha(t+\Delta t) = \alpha(t) + \Delta t$$

and (13)

$$\beta(t+\Delta t) = \beta(t) + \Delta t \quad .$$

Then, from (12) and the fact that coefficients A, B, C and D depend on $(\alpha-\beta)$ only, it can be shown that

$$\phi_1(\alpha+T,\beta+T)-\phi_1(\alpha,\beta) = \exp\left\{\int_\alpha^\beta A(\xi-\beta)d\xi\right\}\exp\left\{i\omega\beta\right\}\left[\exp\left\{i\omega T\right\}-1\right]$$
$$+\int_\beta^\alpha d\xi B(\xi-\beta)\exp\left\{\int_\alpha^\xi A(\xi'-\beta)d\xi'\right\}\int_o^\beta d\eta C(\xi-\eta)\left[\phi_1(\xi+T,\eta+T)-\phi_1(\xi,\eta)\right]\exp\left\{\int_\beta^\eta D(\xi-\eta')d\eta'\right\}$$
$$+\int_{\beta+T}^{\alpha+T} d\xi B(\xi-\beta-T)\exp\left\{\int_{\alpha+T}^{\xi} A(\xi'-\beta-T)d\xi'\right\}\int_o^T d\eta C(\xi-\eta)\phi_1(\xi,\eta)\exp\left\{\int_{\beta+T}^{\eta} D(\xi-\eta')d\eta'\right\} \quad . \tag{14}$$

Now the periodicity assumption is made, i.e. for $t>t^*(x)$, the solution for ϕ_1 will be periodic in time with a period $T=2\pi/\omega$ where $t^*(x)$ is some curve in space such that for any x, if $t<t^*(x)$, ϕ_1 is not periodic. This implies that $t=t^*(x)$ is the dividing curve between regions in which ϕ_1 is periodic and non-periodic. In characteristic coordinates the line $t^*(x)$ is given parametrically as

$$\alpha = t^*(x) + \int_o^x \frac{d\xi}{\sigma(\xi)}$$

$$\beta = t^*(x) - \int_o^x \frac{d\xi}{\lambda(\xi)}$$

which in theory can be written as $\beta = \beta^*(\alpha)$ (see Figure 1). The periodicity assumption then implies that

$$\phi_1(\xi+T,\eta+T) = \phi_1(\xi,\eta) \quad \text{for} \quad \eta > \beta^*(\xi) \ . \tag{15}$$

Then if β is chosen so that $\beta>\beta^*(\alpha)$, which can always be accomplished by choosing $t>t^*(x)$, equation (14) reduces to:

$$\phi_1(\alpha+T,\beta+T) = \phi_1(\alpha,\beta) + G(\alpha,\beta;\beta^*,T) + H(\alpha,\beta;T) \tag{16}$$

where

$$G(\alpha,\beta;\beta^*,T) = \int_\beta^\alpha d\xi B(\xi-\beta)\exp\left\{\int_\alpha^\xi A(\xi'-\beta)d\xi'\right\}$$
$$\times \int_o^{\beta^*(\xi)} d\eta C(\xi-\eta)\left[\phi_1(\xi+T,\eta+T)-\phi_1(\xi,\eta)\right]\exp\left\{\int_\beta^\eta D(\xi-\eta')\ d\eta'\right\} \tag{17}$$

and

$$H(\alpha,\beta;T)=\int_\beta^\alpha d\xi B(\xi-\beta)\exp\left\{\int_\alpha^\xi A(\xi'-\beta)d\xi'\right\} \int_{-T}^{o} d\eta C(\xi-\eta)\phi_1(\xi+T,\eta+T)\exp\left\{\int_\beta^\eta D(\xi-\eta')d\eta'\right\} . \tag{18}$$

Therefore unless (G+H) vanishes the periodicity assumption (15) is contradicted, since then

$$\phi_1(\alpha+T,\beta+T) \neq \phi_1(\alpha,\beta) \quad \text{for} \quad \beta>\beta^*(\alpha) \ . \tag{19}$$

The periodicity of ϕ_1 is now reduced to the question of whether or not (G+H) can vanish. If either C or B vanish, then G=H=0, and the periodicity assumption is not contradicted. Indeed, from (12) it is easy to see that if B or C vanish, then ϕ_1 is periodic. It is also possible that G and H vanish separately without C and B vanishing. But, for H, this would imply that the coefficients of the differential equation depend on $T=2\pi/\omega$, which is specified by the boundary condition, so that H in general does not vanish. In addition, the fact that H does not vanish implies that if $\beta^*(x)=0$ the solution is not periodic, i.e. the solution for vanishing initial conditions cannot be periodic everywhere above the leading characteristic $\beta=0$.

There remains the possibility that G and H do not vanish, but the combination (G+H) does. If (G+H)=0 and if $B \neq 0$ and is independently specified, then from (17) and (18)

$$\int_{-T}^{\beta^*(\xi)} d\eta C(\xi-\eta)\phi_1(\xi+T,\eta+T)\exp\left\{\int_{\beta}^{\eta} D(\xi-\eta')d\eta'\right\} = \int_{o}^{\beta^*(\xi)} d\eta C(\xi-\eta)\phi_1(\xi,\eta)\exp\left\{\int_{\beta}^{\eta} D(\xi-\eta')d\eta'\right\} \tag{20}$$

for $\beta<\xi<\alpha$. Equation (20) may be thought of as a relation that defines for each ξ in the range $\beta<\xi<\alpha$ the number $\beta^*(\xi)$ such that (G+H) vanishes. Note that $\beta^*(\xi)$ will in general depend on T, which is acceptable since one can admit different $\beta^*(\xi)$ values for different choices of $T=2/\pi\omega$. However, from (20) it is obvious that $\beta^*(\xi)$ will also depend on β, i.e. $\beta^*=\beta^*(\xi;\beta)$. In particular, if α is held fixed, and if β is varied, say from $\beta=\beta_1$ to $\beta=\beta_2>\beta_1$, then in general for any ξ such that $\beta_2<\xi<\alpha$

$$\beta^*(\xi;\beta_1) \neq \beta^*(\xi;\beta_2) \quad .$$

This implies that the assumed curve $\beta^*(\alpha)$ [or $t^*(x)$] above which ϕ_1 was assumed to be periodic cannot exist since for any fixed α,β^* depends not only on α but on β which contradicts the assumption that $\beta^*=\beta^*(\alpha)$ only.

In a similar manner, it can be shown that ψ_1 is in general not periodic. By substituting (10) into (11), the integral equation for ψ_1 is

$$\psi_1(\alpha,\beta) = -\int_0^\beta d\eta C(\alpha-\eta)\exp\left\{\int_\beta^\eta D(\alpha-\eta')d\eta'\right\}\left[\exp\left\{-\int_\eta^\alpha A(\xi-\eta)d\xi\right\}\exp(i\omega\eta)\right.$$

$$\left.-\int_\eta^\alpha d\xi B(\xi-\eta)\psi_1(\xi,\eta)\exp\left\{\int_\alpha^\xi A(\xi'-\eta)d\xi'\right\}\right] \tag{21}$$

Then by the same process as for ϕ_1, it can be shown that ψ_1 is not in general periodic. The exceptions are that if C=0, ψ_1 will be periodic. (Note that for B=0, ψ_1 will not be periodic, although ϕ_1 is periodic in this case.)

The conclusion is that unless C vanishes, both ϕ_1 and ψ_1 will not be periodic functions of time with period $2\pi/\omega$. This implies that in general *if there is no initial disturbance, and a boundary disturbance is started at t=0, then the solutions of (6) will not settle into a periodic state.* This is important since vanishing initial conditions are physically applicable.

The problem for ϕ_2 and ψ_2 essentially addresses the question of whether there is an initial condition which, in conjunction with ϕ_1 and ψ_1, will yield a periodic solution. If ϕ_2 and ψ_2 are solved for in a manner similar to ϕ_1 and ψ_1 above, then it can be shown that ϕ_2 and ψ_2 will in general not be periodic. However, one has to consider the possibility that although ϕ_1, ψ_1, ϕ_2, and ψ_2 are not periodic, the combinations $(\phi_1+\phi_2)$ and $(\psi_1+\psi_2)$ are periodic. However, it again can be shown in a manner similar to that for ϕ_1 above that if ϕ_1 and ϕ_2 (and likewise for ψ) are not periodic, then the combination $(\phi_1+\phi_2)$ is periodic only for specially chosen initial data, i.e. the initial conditions will be functions of the boundary condition (namely functions of ω) and/or the coefficients of the differential system. Therefore, if the initial data is arbitrarily specified, then $\phi=\phi_1+\phi_2$ and $\psi=\psi_1+\psi_2$ will not be periodic.

Of course, there do exist initial conditions for which the solution (ϕ,ψ) is periodic. In fact if the elliptic system

$$\begin{pmatrix} \lambda & 0 \\ 0 & -\sigma \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}_x + \begin{pmatrix} \overline{A}+i\omega & \overline{B} \\ \overline{C} & \overline{D}+i\omega \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \qquad (22)$$

with appropriate boundary conditions is well posed, then the initial condition $\phi=a(x)$, $\psi=b(x)$ will yield the periodic solution

$$\begin{pmatrix} \psi \\ \phi \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \exp(i\omega t)$$

Note that since (a,b) are solutions of (22), they depend on ω and the coefficients of the differential system (1). There even may be other specially chosen initial conditions which will yield periodic solutions for t large enough. However, it is easy to show that if such a periodic solution does exist, then if the initial data, the coefficients of the differential equation, or the frequency or amplitude of the boundary data are perturbed, the resulting solution will not be periodic. For example, if (ϕ,ψ) is a periodic solution of (1) with

$$(\phi,\psi)\big|_{t=0} = (f,g) \qquad \text{and} \qquad \phi\big|_{x=0} = e^{i\omega t}$$

for some specially chosen (f,g), then the solution of (1) with

$$(\overline{\phi},\overline{\psi})_{t=0} = (f,g) \qquad \text{and} \qquad \overline{\phi}\big|_{x=0} = (1+\varepsilon)e^{i\omega t}$$

will not be periodic since the combination

$$(\phi',\psi') = \frac{1}{\varepsilon}\,[(\overline{\phi},\overline{\psi})-(\phi,\psi)]$$

satisfies (1),

$$(\phi',\psi')|_{t=0} = 0 \qquad \text{and} \qquad \phi'|_{x=0} = e^{i\omega t}$$

which has been shown above to possess a non-periodic solution.

All the other side conditions discussed in section 2 yield similar results to those for the case treated above. The results are summarized in the following list:

1. If the coefficients B and C in (6) do not vanish, then the solution of (6) with arbitrary initial data, and periodic boundary and/or source conditions will in general not settle into a periodic state;

2. Statement 1 remains true even for the particular case of vanishing initial data;

3. It is possible to construct special initial data which yield periodic solutions. However, this special data will depend on the boundary and/or source data, as well as on the coefficients of the differential system. Moreover, any perturbation to the governing system with the special initial data will result in a non-periodic perturbation to the solution;

4. If the coefficients B and C in (6) vanish, and the initial data vanish, then the solution of (6) will become periodic. In fact, if the initial data is non-zero in a compact region, the solution will eventually be periodic also, and if the initial data asymptotically vanishes as $|x|\to\infty$, then the solution will also be asymptotically periodic for $t\to\infty$.

4. SYSTEMS WITH SMALL LOWER ORDER COEFFICIENTS

Suppose that the coefficients $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$ in (1) do not identically vanish, but are small in the sense that

$\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D} = o(1)$.

Then the system (1) can be written in the form

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}_t + \begin{pmatrix}\lambda & 0\\ 0 & -\sigma\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_x + \varepsilon\begin{pmatrix}\overline{A} & \overline{B}\\ \overline{C} & \overline{D}\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0 \tag{23}$$

where $\varepsilon \ll 1$ and now $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D} = O(1)$. With side conditions (which are in general independent of ε), the solution of (23) may be obtained by a regular perturbation scheme by formally seeking solutions of the form

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}\phi^o\\ \psi^o\end{pmatrix} + \varepsilon\begin{pmatrix}\phi^1\\ \psi^1\end{pmatrix} + \varepsilon^2\begin{pmatrix}\phi^2\\ \phi^2\end{pmatrix} + \ldots \quad .$$

The governing system for (ϕ^o, ψ^o) is (in characteristic variables) simply

$$\phi^o_\alpha = 0 \quad \text{and} \quad \phi^o_\beta = 0 \tag{24}$$

which according to statement 4 at the end of section 3 has a periodic solution for vanishing initial data. In fact, from (12) and (21) if $\phi(x=0)=\exp(i\omega t)$ then

$$\phi^o(\alpha,\beta) = \exp(i\omega\beta)$$

and

$$\psi^o(\alpha,\beta) = 0$$

above the right running characteristic passing through the origin.

The governing system for (ϕ^1, ψ^1) is then

$$\phi^1_\alpha = -A(\alpha-\beta)\exp(i\omega\beta)$$

$$\psi^1_\beta = -C(\alpha-\beta)\exp(i\omega\beta)$$

with $\phi^1(\beta,\beta)=0$ and $\psi^1(\alpha,0)=0$. Then

$$\phi^1 = -\exp(i\omega\beta) \int_\beta^\alpha A(\xi-\beta)d\xi$$

$$\psi^1 = -\int_0^\beta C(\alpha-\eta)\exp(i\omega\eta)d\eta \quad .$$

Note that ϕ^1 is periodic, i.e. $\phi^1(\alpha+T,\beta+T)=\phi^1(\alpha,\beta)$, but ψ^1 is not. In fact

$$\psi^1(\alpha+T,\beta+T) = \psi^1(\alpha,\beta) + \int_0^T C(\alpha+T-\eta)\exp(i\omega\eta)d\eta.$$

If the process is continued, the solutions for ϕ^2 and ψ^2 will not be periodic. Therefore we are led to the following results for the boundary value problem with vanishing initial data:

$$\phi = \text{periodic function} + \varepsilon^2(\text{non-periodic function})$$

$$\psi = \text{periodic function} + \varepsilon\ (\text{non-periodic function}) \ .$$

For problems driven by a periodic source with vanishing initial data, both ϕ and ψ will be non-periodic starting with the term proportional to ε. In general, a problem governed by a system such as (23) has a solution whose leading term is periodic. In all cases this periodic term may be easily found by ray tracing or following characteristics by solving the system (24).

It seems that if the coefficients of the undifferentiated terms of (1) are small, one could conclude that the dominant term of the solution is periodic, and that the non-periodicity is contained in a small perturbation term. However, it can easily be shown that this series solution in ε is not necessarily uniformly valid in time, i.e. $\varepsilon\phi^1$ and/or $\varepsilon\psi^1$ can grow in time and become comparable to or dominate over the zeroth order term. Therefore, only for those coefficients for which the non-periodic part of the solution does not exhibit such growth, i.e. for ϕ^1 and ψ^1 such that

$$\varepsilon\phi^1 = o(\phi^o) \quad , \quad \varepsilon\psi^1 = o(\psi^o) \quad ,$$

will the periodic part of the solution dominate over the non-periodic part. It is usually the case that the signs of combinations of the coefficients A, B, C and D determine whether or not ϕ^o and ψ^o do indeed dominate over $\varepsilon\phi^1$ and $\varepsilon\psi^1$, respectively.

5. COMPACT LOWER ORDER COEFFICIENTS

A very important class of problems are those for which the coefficients A, B, C and D of (6) either vanish everywhere except for a bounded region or decay asymptotically to zero as the distance from the origin increases.

Suppose these coefficients vanish except for a region near the origin $x<x_o$. In particular, for $x>x_o$ C=0 so that from (11) or (21)

$$\psi = 0 \text{ for } x > x_o \quad . \tag{25}$$

for vanishing initial data.

In Figure 2, the point marked 1 is the intersection of the characteristic curve α = constant with the line $x=x_o$. The characteristic coordinates of this point are

$$\alpha_1 = \alpha \text{ and } \beta_1 = \alpha - d(x_o)$$

where

$$d(x_o) = \int_o^{x_o} d\xi[1/\lambda(\xi) + 1/\sigma(\xi)]. \tag{26}$$

Then from (11) and C=0 for $x>x_o$,

$$\psi(\alpha,\beta) = -\int_{\beta_1(\alpha)}^{\beta} d\eta C(\alpha-\eta)\phi(\alpha,\eta)\exp\left\{\int_{\beta}^{\eta} D(\alpha-\eta')d\eta'\right\} \tag{27}$$

for $x<x_o$ and t large enough.

In Figure 3, the point marked 2 is the intersection of the characteristic curve β = constant with the line $x=x_o$. The characteristic coordinates of this point are

$$\beta_2 = \beta \quad \text{and} \quad \alpha_2 = \beta + d(x_o) \ . \tag{28}$$

Then, from (10) and A=B=0 for $x>x_o$:

$$\phi(\alpha,\beta)=\exp\left\{ -\int_{\beta}^{\alpha_2(\beta)} A(\xi-\beta)d\xi\right\}\exp\left\{i\omega\beta\right\} -\int_{\beta}^{\alpha_2(\beta)} d\xi B(\xi-\beta)\psi(\xi,\beta)\exp\left\{\int_{\alpha_2(\beta)}^{\xi} A(\xi'-\beta)d\xi'\right\} \tag{29}$$

for $x>x_o$. For $x<x_o$, ϕ is given by (10).

The expression (29) for ϕ includes an integrand which depends on values of ψ for $x<x_o$. Then, by substituting (27) into (29)

$$\begin{aligned}\phi(\alpha,\beta)=&\exp -\left\{\int_{\beta}^{\alpha_2(\beta)} A(\xi-\beta)d\xi\right\} \exp\left\{i\omega\beta\right\}\\ &+\int_{\beta}^{\alpha_2(\beta)} d\xi B(\xi-\beta)\exp\left\{\int_{\alpha_2(\beta)}^{\xi} A(\xi'-\beta)d\xi'\right\}\int_{\beta_1(\xi)}^{\beta} d\eta C(\xi-\eta)\phi(\xi,\eta)\exp\left\{\int_{\xi}^{\eta} D(\xi-\eta')\ d\eta'\right\}\end{aligned} \tag{30}$$

for $x>x_o$ and t large enough.

From (26) and (28)

$$\beta_1(\alpha+T) = \beta_1(\alpha) + T$$

and

$$\alpha_2(\beta+T) = \alpha_2(\beta) + T \ .$$

Then from (30)

$$[\phi(\alpha+T,\beta+T)-\phi(\alpha,\beta)]=\int_{\beta}^{\alpha_2(\beta)} d\xi B(\xi-\beta)\exp\left\{\int_{\alpha_2(\beta)}^{\xi} A(\xi'-\beta)d\xi'\right\}$$

$$\times\int_{\beta_1(\xi)}^{\beta} d\eta C(\xi-\eta)[\phi(\xi+T,\eta+T)-\phi(\xi,\eta)] \quad \exp\left\{\int_{\beta}^{\eta} D(\xi-\eta')d\eta'\right\} \tag{31}$$

which is a homogeneous integral equation of the second kind. Therefore

$$\phi(\alpha+T,\beta+T) = \phi(\alpha,\beta)$$

for $x>x_o$ and t large enough. This in turn implies that ϕ is periodic in time with a period $T=2\pi/\omega$. Then (25) and (31) imply that the complete solution ϕ and ψ of (6) is periodic in time with a period T.

In a similar manner, it can be shown that for $x<x_o$ both ϕ and ψ are periodic in time with a period T for large enough values of t. For both $x<x_o$ and $x>x_o$, large enough time means that the point of intersection of characteristic curves $\eta = 0$ and $\xi = \beta$ (which is the point marked 3 in Figures 2 and 3) lies to the right of the line $x=x_o$. To accomplish this for a given x, one only has to choose

$$t \geq \int_{o}^{x} \frac{d\xi}{\lambda(\xi)} + d(x_o) \quad . \tag{32}$$

For t not satisfying (32), non-periodicity of the acoustic field can be shown in a similar manner to that used in section 3.

Now consider the case of inhomogeneities that asymptotically vanish as x increases. It is easy to see that for large enough t, ϕ will be asymptotically periodic. The last term of (14) will then vanish because it then depends on B and C evaluated for large values of x only. The shaded region in Figure 4 is the region of integration for the last term of (14), and for any x, as t increases this region moves to the right so that the values of B and C in this region asymptotically vanish. Once the last term of (14) becomes very small, (14) becomes similar to (31). Therefore, for any x, ϕ is asymptotically periodic for large t. Likewise, it can be shown that for any x, ψ is asymptotically periodic for t large enough.

To summarize, it has been shown that if the coefficients A, B, C and D vanish everywhere except in a region near the origin, then for any x the solution field will be periodic if t satisfies (32). Moreover, if these coefficients vanish asymptotically for large $|x|$, then the solution field will become asymptotically periodic for large t. Similar conclusions can be made for the other boundary and/or source conditions discussed in section 2.

6. IMPLICATIONS FOR PHYSICAL PHENOMENA

The propagation of disturbances through nonuniform media are often governed by hyperbolic systems of partial differential equations such as (1). Typically, the coefficients λ and σ in (1) depend on properties of the media such as the speed of sound and if the media is a fluid, the fluid velocity. The coefficients $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ depend not only on the media properties, but also on the gradients of these. In particular, these coefficients vanish with those gradients. It is often, but not always, the case that the linearity of the system of partial differential equations is a result of assuming that the propagating disturbances are small compared to the corresponding media properties. In any case, the linearity of the system implies that the propagating disturbance does not affect the basic media properties that enter into the coefficients of the system.

Problems dealing with the propagation of waves which are governed by systems such as (1) arise in numerous branches of physics. The periodicity assumption is a tool that is almost universally used to simplify the analysis of such phenomena. Examples of areas where systems of equations similar to (1) arise are, among others, the study of wave propagation through the atmosphere [1], the ocean [2], electric and magnetic fields [3], layered media [4], plasmas [5], elastic bars [3], etc. In all these cases, extensive use is made of the periodicity assumption. Based on the results of sections 3, 4, and 5, we can reach some general conclusions about the periodicity of disturbances and thus about the validity of the periodicity assumption. These conclusions apply to any field in which the fluctuations are governed by systems such as (1) and which are driven by periodic sources or boundary conditions.

1. If the non-uniformities of the media extend throughout space and have gradients which are not "small", the propagating fluctuations will in general not be periodic;

2. If the gradients of the media properties are "small" (compared to the properties themselves), then in general the fluctuations will not be periodic. However, there are situations in which the fluctuations are almost periodic in the sense that non-periodic effects are small compared to a dominant periodic fluctuation. It is usually the case that the signs of the gradients of the media properties determine whether or not the periodic fluctuations are indeed dominant for all time;

3. If the gradients of the media properties and the prescribed initial conditions vanish everywhere except for a compact region of space, then the propagating fluctuations will, after some time, become periodic. There is no restriction on the magnitude of these gradients in the region where they do not

identically vanish. Furthermore, if these gradients and initial data vanish asymptotically at large distances, then the fluctuations will become asymptotically periodic for increasing time.

We now consider in greater detail one particular physical situation where the above conclusions apply.

Propagation of pressure fluctuations through steady flows of lossless fluids

We first show that a system such as (1) may indeed govern the propagation of pressure fluctuations through one dimensional steady flow. All model problems considered neglect any effects due to viscosity and heat conduction. For simplicity, it is assumed that the media is a perfect gas, although this assumption is by no means necessary. Throughout the mean flow will be denoted by subscript o, and the acoustic perturbation by variables without subscript. The pressure is denoted by p, the density by ρ, the speed of sound by c, and the velocity by u.

For a one dimensional moving inhomogeneous medium (with area change), the equations governing the propagation of sound are given by:

$$\frac{\partial p}{\partial t} + \rho_o c_o^2 \frac{\partial u}{\partial x} + u_o \frac{\partial p}{\partial x} + [u \frac{\partial p_o}{\partial x}] + [\gamma \frac{\partial u_o}{\partial x} p] + [p\gamma u_o \frac{d}{dx}(\ln S) + u\rho_o c_o^2 \frac{d}{dx}(\ln S)] = 0 \tag{33}$$

and

$$\frac{\partial u}{\partial t} + \frac{1}{\rho_o} \frac{\partial p}{\partial x} + u_o \frac{\partial u}{\partial x} - [\frac{p}{\rho_o c_o^2} \frac{\partial p_o}{\partial x}] + [u \frac{\partial u_o}{\partial x}] = 0$$

where γ is the constant ratio of specific heats and $S=S(x)$ is the cross-sectional area. In each equation the first bracketed term is due to the pressure gradient of the mean flow, the second is due to the velocity gradient of the mean flow, and the third is due to the change in cross-sectional area. Equation (33) is valid so long as the mean flow is time independent and the source or boundary disturbance is acoustic in nature, e.g. is a pressure source and not an entropy source.

The system (33) can be simplified by imposing restrictions on the mean flow. For instance, if the medium is at rest with constant pressure (u_o=0, p_o=constant) and the cross-sectional area is constant (S=constant), only the first two terms in each equation fail to vanish. In fact, the two equations may be combined to yield the familiar

$$\frac{\partial^2 p}{\partial t^2} - \rho_o c_o^2 \frac{\partial}{\partial x} \left(\frac{1}{\rho o} \frac{\partial p}{\partial x}\right) = 0 .$$

Other simpler systems of interest are ones for flows at rest with constant entropy (u_o=0, $p_o \rho_o^{-\gamma}$=constant) and flows with no area change (S=constant).

In the characteristic variables

$$\phi = p + \rho_o c_o u$$

and (34)

$$\psi = p - \rho_o c_o u$$

the system for u_o=0, p_o=constant and S=constant becomes

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}_t + c_o \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_x + \frac{1}{2}\frac{\partial c_o}{\partial x} \begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0 \qquad (35)$$

For u_o=0, S=constant and constant entropy, the characteristic system is

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}_t + c_o \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_x + \frac{1}{2(\gamma-1)}\frac{\partial c_o}{\partial x}\begin{pmatrix}-\gamma-1 & \gamma-3\\ 3-\gamma & \gamma+1\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0\ . \tag{36}$$

For the full system (35), the characteristic system is

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}_t + \begin{pmatrix}u_o+c_o & 0\\ 0 & u_o-c_o\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_x + \frac{1}{2}\left\{\frac{1}{(\gamma-1)}\frac{\partial c_o}{\partial x}\begin{pmatrix}-\gamma-1 & \gamma-3\\ 3-\gamma & \gamma+1\end{pmatrix} + \left(\frac{\gamma+1}{\gamma-1}\right)\frac{u_o}{c_o}\frac{\partial c_o}{\partial x}\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}\right.$$

$$\left. + \frac{\partial u_o}{\partial x}\begin{pmatrix}\gamma+1 & \gamma-1\\ \gamma-1 & \gamma+1\end{pmatrix} + \frac{d}{dx}(\ln S)\begin{pmatrix}\gamma u_o+c_o & \gamma u_o-c_o\\ \gamma u_o+c_o & \gamma u_o-c_o\end{pmatrix}\right\}\begin{pmatrix}\phi\\ \psi\end{pmatrix} = 0\ . \tag{37}$$

For any other specialized mean flow, an appropriate system can be found from (37). It is obvious that all three systems (35), (36) and (37) are of the same form as (1).

The important thing to note is not the differences in the systems (35) - (37), but rather their similarities. The most important similarity is that for the characteristic dependent variables, all systems will have a term involving undifferentiated values of ϕ and ψ . This is true even for the extremely simple case of system (35). The only exception is when all gradients vanish, i.e., u_o, c_o and S are constant, in which case (37) becomes

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix}_t + \begin{pmatrix}u_o+c_o & 0\\ 0 & u_o-c_o\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}_x = 0 \tag{38}$$

regardless of whether $u_o=0$ or u_o=constant. Therefore, the essential simplification is not $u_o=0$, but rather zero gradients since it was seen in section 3 that it is precisely those terms which appear in (35), (36), or (37) and not in (38) which can bring about the non-periodicity of the solution. It is also clear that any restrictions made on the coefficients $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$ in (1) are from (37) restrictions on the gradients of the media properties c_o, u_o and S. In particular, if $\bar{A}$, etc. vanish everywhere except for a bounded region of space then it must be that the fluid is uniform and is moving at a constant velocity everywhere except for a bounded region of space. A similar analogy holds for the case of $\bar{A}$, etc., vanishing asymptotically for large distances. This last situation is of particular importance since in many physical situations, such as jets (see [6,7]), the media non-uniformities do indeed decay in such a manner.

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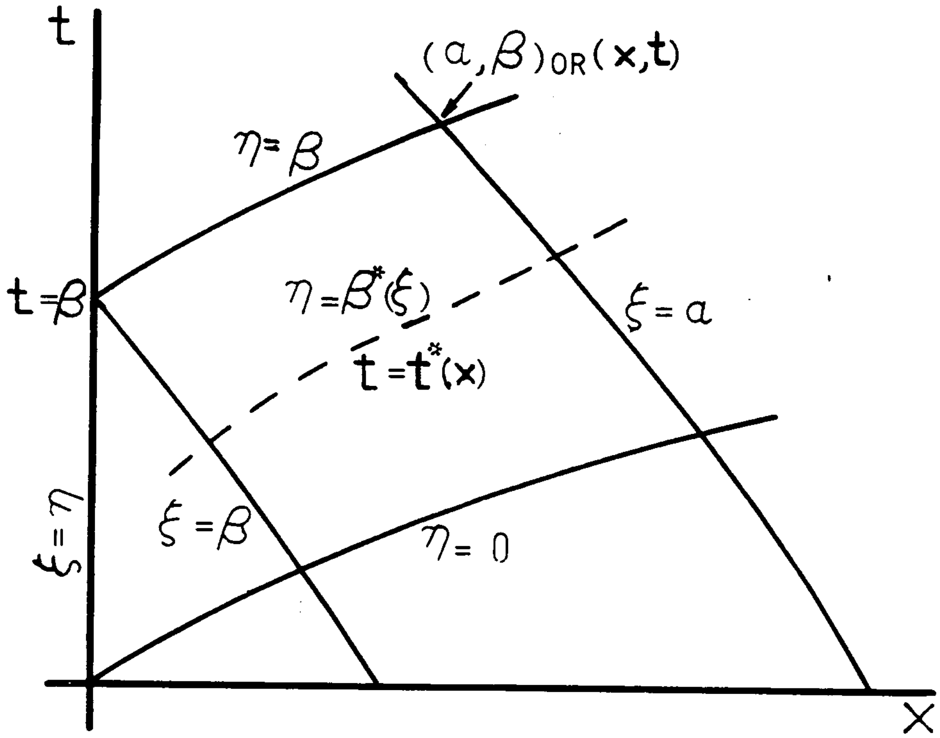

Figure 1. Characteristic geometry and the curve t=t*(x).

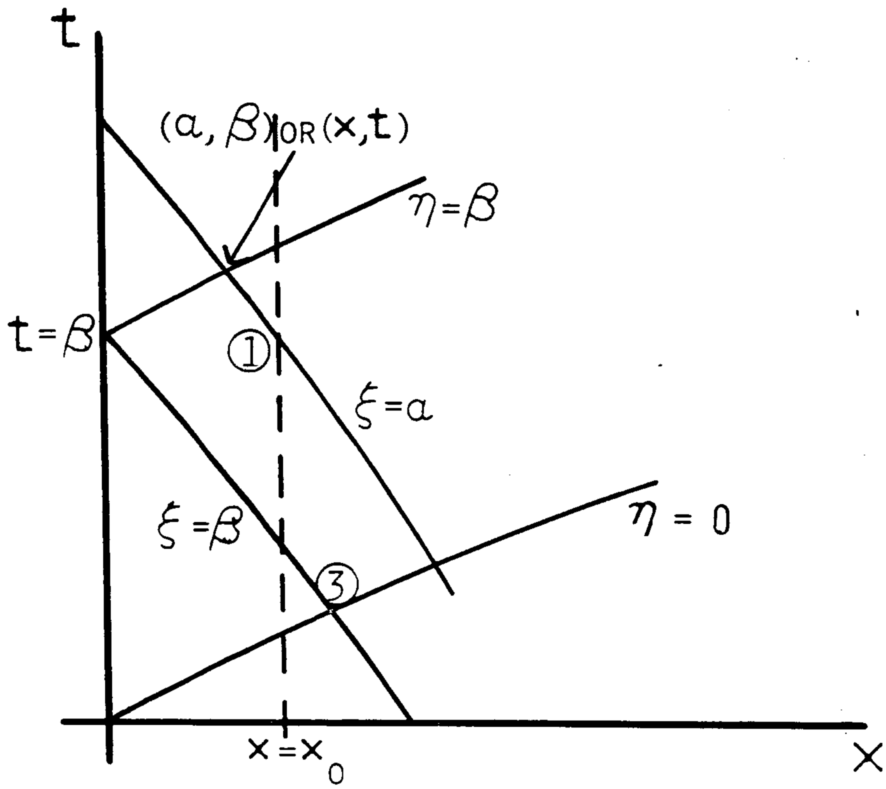

Figure 2. Characteristic geometry for (x,t) such that $x<x_o$.

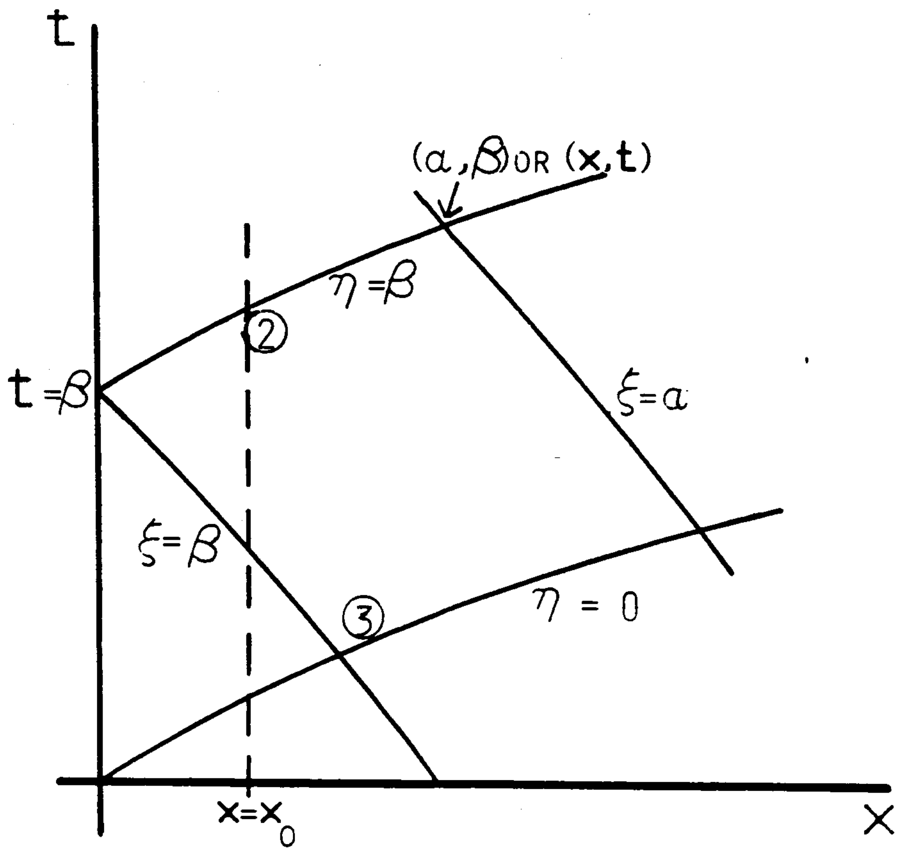

Figure 3. Characteristic geometry for (x,t) such that $x>x_o$.

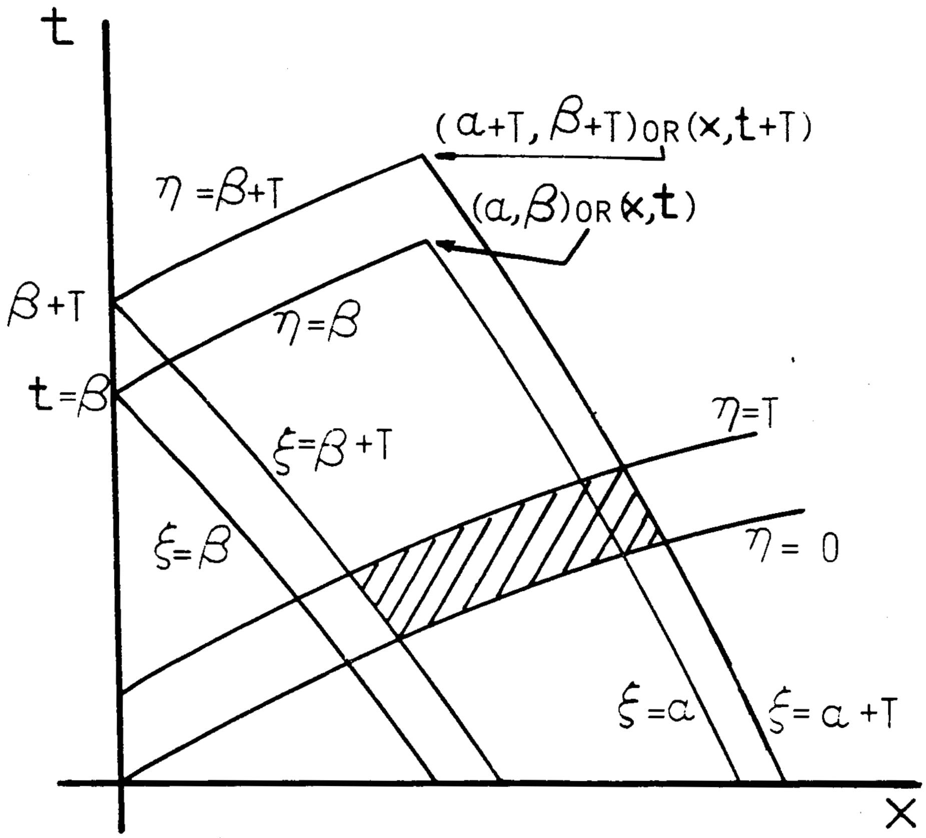

Figure 4. Regions of integration for points one period apart.